

KOSZUL DUALITY IN ALGEBRAIC TOPOLOGY

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The most prevalent examples of Koszul duality of operads are the self-duality of the associative operad and the duality between the Lie and commutative operads. At the level of algebras and coalgebras, the former duality was first noticed as such by Moore, as announced in his ICM talk at Nice [13]. Thus this particular duality has typically been called Moore duality, and some prefer to call the general phenomenon Koszul-Moore duality. The second duality at the level of algebras was realized in the seminal work of Quillen on rational homotopy theory [15]. Our aim in these notes based on our talk at the Luminy workshop on Operads in 2009 is to try to provide some historical, topological context for these two classical algebraic dualities.

We first review the original cobar and bar constructions used to study loop spaces and classifying spaces, emphasizing the less-familiar geometry of the cobar construction. Then, after some elementary topology, we state duality between bar and cobar complexes in that setting. Before explaining Quillen's work, we also share some other ideas - calculations of Cartan-Serre and Milnor-Moore and philosophy of Eckmann-Hilton - which may have influenced him. After stating Quillen's duality, we share some recent work which relates these constructions to geometry through Hopf invariants and in particular linking phenomena.

I would like to thank my collaborator Ben Walter. This material is a union of standard material which either I taught him or he taught me along with new theorems which we have figured out together.

1. BAR AND COBAR CONSTRUCTIONS

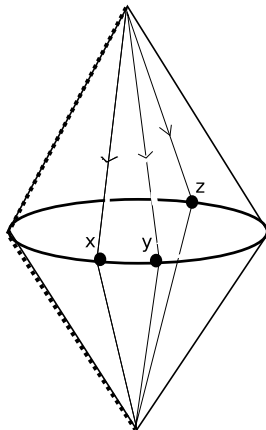
1.1. ΩX and the cobar construction. Studying mapping spaces is one of the central tasks of topology, and loop spaces are the simplest and most fundamental examples (unless one counts maps from finite sets, which yield products). We require a model for loops where the loop sum is associative exactly, not up to homotopy. Thus, for us ΩX denotes the Moore loop space which consists of pairs $f : \mathbb{R} \rightarrow X$ and a "curfew" $a > 0$ such that $f(x)$ is the basepoint if $x \leq 0$ or if $x \geq a$. Loop sum adds these curfews, which makes multiplication associative.

The cobar construction of Adams and Hilton [2] was informed by the almost concurrent work of James [10] who studied $\Omega\Sigma X$, the loop space on the reduced suspension of X , namely $\Sigma X = X \times \mathbb{I}/(X \times 0 \sim * \times \mathbb{I} \sim 1 \times X)$. There is a canonical inclusion of $J : X \hookrightarrow \Omega\Sigma X$ sending x to $J(x)(t)$, the path which sends t to the image in ΣX of (x, t) . Because $\Omega\Sigma X$ is a topological monoid, this map extends to a map from the free monoid (with unit) on X to $\Omega\Sigma X$ which we call the James map \hat{J} . For example, the formal product $y * x * z$ goes to a loop with coordinates (x, t) for $t \in [0, 1]$ then $(y, t - 1)$ for $t \in [1, 2]$, then $(z, t - 2)$ for $t \in [2, 3]$ - see the figure below.

Theorem 1.1 (James [10]). *The James map \hat{J} from the free monoid on X to $\Omega\Sigma X$ is a homotopy equivalence.*

Recall that the homology of any space with an associative multiplication, or even a homotopy associative multiplication, is an associative algebra.

Corollary 1.2. *The homology of $\Omega\Sigma X$ with field coefficient is isomorphic as an algebra to the tensor (that is, free associative) algebra on the homology of X .*



An illustration of \hat{J} of $y * x * z$ (traversing the path through y first, etc).

Exercise 1. *Explicitly define the free topological monoid on a (well-based) topological space X . Show that its homology with field coefficients is isomorphic to the tensor algebra on the homology of X . [Hint: make heavy use of the Künneth theorem.]*

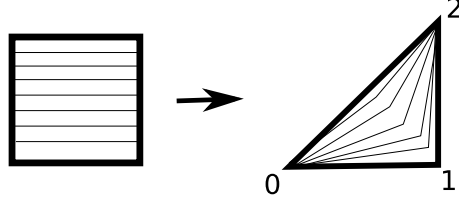
Interestingly, the corollary is typically proven in the course of proof of the theorem. Details in a well-digested form are in Section 4.J in Hatcher's textbook [9] or the survey paper of Carlsson and Milgram [4], whose treatment of the Adams-Hilton construction heavily influences our treatment below. There are however more geometric proofs which build on the fact that the space of paths in the cone on X with endpoints in the image of X is homotopy equivalent to $X \times X$ through the projection onto the endpoints.

For the Adams-Hilton construction, we start with a simply-connected simplicial complex \hat{X} and then contract the union of the 1-skeleton along with enough of the two-skeleton so that the quotient map $\hat{X} \rightarrow X$ is a homotopy equivalence. Then X is a CW-complex, and its cellular chains are a quotient of the simplicial chains on \hat{X} . By abuse of notation, we denote these cellular chains by $C_*^\Delta(X)$.

Next, consider the cubical singular chain complex of the loop space $C_*^\square(\Omega X)$, which is an associative differential graded algebra. On generators, the product of $\sigma_1 : \mathbb{I}^n \rightarrow \Omega X$ and $\sigma_2 : \mathbb{I}^m \rightarrow \Omega X$ is the composite $\mathbb{I}^{n+m} \cong \mathbb{I}^n \times \mathbb{I}^m \xrightarrow{\sigma_1 \times \sigma_2} \Omega X \times \Omega X \rightarrow \Omega X$.

The Adams-Hilton construction defines a map of associative algebras from the free associative algebra on $C_*^\Delta(X)$ to $C_*^\square(\Omega X)$. The first key observation is that any choice of map $\gamma_n : \mathbb{I}^n \rightarrow \Delta^n$ defines a map $AH_{\gamma_n} : C_n^\Delta(X) \rightarrow C_{n-1}^\square(\Omega X)$. Let χ_σ denote the characteristic map $\Delta^n \rightarrow X$ of a simplex σ of X . Then $AH_{\gamma_n}(\sigma)$ is basically given by the composite $\gamma_n \circ \chi_\sigma : \mathbb{I}^n \rightarrow X$. From this composite we by adjointness (choosing say the last coordinate as the loop coordinate) produce a map $\mathbb{I}^{n-1} \rightarrow \text{Map}(\mathbb{I}, X)$, which then is identified with a generator of $C_{n-1}^\square(\Omega X)$ through viewing $\text{Map}(\mathbb{I}, X)$ as Moore loops with curfew one.

The game is to define γ_n appropriately so that we can calculate boundaries, and more importantly so that the Adams-Hilton map yields a quasi-isomorphism. By abuse, we suppress γ_n from notation and write $AH_{\gamma_n}(\sigma)$ as $|\sigma|$. For the first case when $n = 2$, a good way to choose γ_2 is to have $\gamma_2 : \mathbb{I}^2 \rightarrow \Delta^2$ send the boundary of \mathbb{I}^2 to that of Δ^2 . In any way this is done, we would have that $d|\sigma_2| = |d\sigma_2| = 0$, since the one-skeleton of X has been identified to a point. Looking forward, it is much better to choose γ_2 to be a “degree one” map $\mathbb{I}^2 \rightarrow \Delta^2$ which when we consider the adjoint $\hat{\gamma}_2 : \mathbb{I} \rightarrow \text{Map}(\mathbb{I}, \Delta^2)$ interpolates between the direct path from vertex 0 to vertex 2 of Δ^2 along the edge between them and the “long” path from 0 to 2 which first traverses the 0-1-edge and then the 1-2 edge. When composed with the characteristic map into X these edge paths will yield constant loops, but the choice of the paths in between is important.

One possible choice for γ_2 .

At the next stage, building on some such choice of degree one γ_2 , we can define a γ_3 such that $d|\sigma_3| = |d\sigma_3|$. There are four faces of \mathbb{I}^2 and four faces of Δ^3 , and this equality identifies those faces. For example, one face of \mathbb{I}^2 will be mapped by $\hat{\gamma}_3 : \mathbb{I}^2 \rightarrow \text{Map}(\mathbb{I}, \Delta^3)$ to paths from the 0 vertex to the vertex 3 (reminder: such a path in \hat{X} will project to a loop in X) which first go to the vertex 1 directly along the 0-1 edge and then go to 3 along paths compatible with the choice made of γ_2 . On another face of \mathbb{I}^2 , paths go only along the 0-2-3 face of Δ^3 , again compatibly with γ_2 , and so on.

At $n = 4$ the construction there is a surprise. Assume γ_3 has been defined, and start defining $\hat{\gamma}_4 : \mathbb{I}^3 \rightarrow \Delta^4$ by setting its restriction to various faces as before, for example sending one face of \mathbb{I}^3 to paths on the 0-2-3-4 face of Δ^4 . But, there are six faces of \mathbb{I}^3 and only five faces of Δ^4 ! What is the natural last term? When one does the geometry carefully, one see that on the last face of \mathbb{I}^3 should map to paths from 0 to 4 which first go along the 0-1-2 face and then along the 2-3-4 face. These two faces appear in the standard definition of the coproduct on simplicial chains dual to cup product. Moreover, such composites are given by the product in $C_*^\square(\Omega X)$. That is, we can construct γ_4 such that

$$d|\sigma_4| = |d\sigma_4| + |\alpha_2| * |\beta_2|,$$

where the coproduct of σ_4 is $\alpha_2 \otimes \beta_2$ plus terms in bidegrees $(1, 3)$, $(0, 4)$, etc. (These other terms in the coproduct yield trivial chains with the 1-skeleton of X collapsed.)

In general, let us denote products of Adams-Hilton chains by $|\sigma| * |\tau| = |\sigma|\tau|$. Let $\text{Cobar}(C_*^\Delta(X))$ denote the sub-algebra of $C_*^\square(\Omega X)$ generated by the Adams-Hilton chains (in positive degrees).

Theorem 1.3 (Adams-Hilton). *There are degree-one choices for the maps γ_n such that the boundary on $\text{Cobar}(C_*^\Delta(X))$ is the cofree extension of the map with*

$$d|\sigma| = \pm |d\sigma| + \sum_{\bar{\Delta}\sigma = \sum \alpha_i \otimes \beta_i} \pm |\alpha_i| |\beta_i|.$$

Here $\bar{\Delta}$ denotes the reduced cup coproduct including terms of only positive bidegrees.

The inclusion of any such $\text{Cobar}(C_*^\Delta(X))$ in $C_*^\square(\Omega X)$ is a quasi-isomorphism of differential graded associative algebras.

Exercise 2. Try to write down γ_n for $n \leq 4$ as an explicit piecewise-linear map.

Exercise 3. The cobar construction is defined for any differential graded coalgebra. Compute it for the coalgebra given by the homology of CP^∞ .

Exercise 4. Deduce the James theorem from the Adams-Hilton theorem.

The algebraic cobar construction (often denoted Ω but not at the moment because of potential confusion) has become part of the standard toolkit for algebraic topologists, and there are more algebraic approaches which can yield similar theorems. A more geometric approach to the topology of iterated loop spaces was extended by Milgram who studied $\Omega^n \Sigma^n X$ in [11] (see also [4]). But the geometry and formalism of PROPs and operads, in particular the elegance of the little disks construction of Boardman and Vogt [3], became more popular than this intricate geometry. Perhaps there could be something gained by revisiting these ideas.

1.2. Classifying spaces and the bar construction. We will be more brief about the bar construction, whose topology is better known. The topological bar construction provides a model for the classifying space BG , which when G is discrete is just the Eilenberg-MacLane space $K(G, 1)$.

Topologists are often ambiguous and refer to any quotient of a contractible space with free G -action as its classifying space BG . We resolve this issue by only saying that a space is homotopy equivalent (rather than equal to) BG . As we remark below, there is a choice which is useful to call BG unambiguously.

Example 1.4. • $B\mathbb{Z} \simeq S^1$.

- $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$.
- $B\mathbb{Z}/n \simeq S^\infty/(\mathbb{Z}/n)$, called an infinite Lens space.
- If $G = \pi_1(S)$ where S is a surface of positive genus, then $BG \simeq S$.

Theorem 1.5. *If G is discrete, then BG is homotopy equivalent to a simplicial complex whose n -simplices are in one-to-one correspondence with n -tuples of elements of G , which we denote $|g_1|g_2|\cdots|g_n|$. The $(n+1)$ faces of an n -simplex are given by*

$$d_i(|g_1|\cdots|g_n|) = \begin{cases} |g_2|\cdots|g_n| & i = 0 \\ |g_1|\cdots|g_i g_{i+1}|\cdots|g_n| & 0 < i < n \\ |g_1|\cdots|g_{n-1}| & i = n. \end{cases}$$

To prove this, one constructs EG in a similar fashion.

Corollary 1.6. *The homology of BG is given by the homology of the algebraic bar construction applied to the group ring $\mathbf{k}[G]$, an associative algebra.*

Exercise 5. *Do the simple unraveling of definitions to check that this corollary follows.*

We obtain a better model if we quotient by identifying each n -simplex of the form $|g_1|\cdots|e|g_{i+1}|\cdots|g_n|$ with the $(n-1)$ -simplex $|g_1|\cdots|g_{i-1}|g_{i+1}|\cdots|g_n|$ through the appropriate standard projection of $\Delta^n \rightarrow \Delta^{n-1}$. The following exercise is a must for any topology student.

Exercise 6. *Show that this reduced construction for $\mathbb{Z}/2$ is homeomorphic to $\mathbb{R}P^\infty$.*

Thus $\mathbb{R}P^\infty$ has $\mathbb{Z}/2$ as its DNA, so to speak. Theorem 1.5 is true in greater generality in particular when G has a topology (with some mild assumptions) which gets incorporated in the topology on BG , or when G is just a monoid. Indeed, this construction is a special case of the nerve of a category.

1.3. Relating the bar and cobar constructions. We defined the homotopy type of BG through the fiber sequence

$$G \subset EG \rightarrow BG.$$

Let PX denote the path space on X , which is contractible, and let ev denote the map which sends a path γ to $\gamma(1) \in X$. Then the sequence

$$\Omega X \rightarrow PX \xrightarrow{ev} X$$

is a fibration. Consider as well the map $PEG \rightarrow BG$ defined by evaluation composed by the quotient. This map is equivalent to both the projection $EG \rightarrow BG$ and the evaluation $PBG \rightarrow BG$, which are thus equivalent to each other. We deduce that their fibers are equivalent, so that $\Omega BG \simeq G$. Similarly, if X is connected then $B\Omega X \simeq X$ (the content of this statement depends on the definition of classifying space for ΩX ; some say its classifying space is X by definition).

These homotopy equivalences are reflected in the following algebra, which is now viewed as a consequence of Koszul duality of the associative operad. Recall that the cobar construction was defined in terms of a free associative algebra (and indeed computed the homology of ΩX as an algebra). We can view the bar construction as based on the free coassociative coalgebra generated by $\mathbf{k}[G]$, with the coproduct defined by breaking bar expressions in two and differential defined using the product of G .

Theorem 1.7. *The bar construction B and the cobar construction Ω define an adjoint pair of functors between differential graded associative algebras DGAA and differential graded associative coalgebras DGAC.*

$$\text{DGAC} \begin{array}{c} \xrightarrow{\Omega} \\ \xleftarrow{B} \end{array} \text{DGAA}$$

Moreover, there are natural transformations $\Omega BA \rightarrow A$ and $B\Omega C \rightarrow C$ which are quasi-isomorphisms if A is positively graded and if C is 1-connected respectively.

This theorem was announced by Moore [13], so it has historically been referred to as Moore duality. In topology, this equivalence reflects the bijection between homotopy classes of monoid maps from some M to ΩX and homotopy classes of maps from BM to X .

Not only is it the first example of adjoint functors giving equivalences between categories of algebras and coalgebras over an operad and its Koszul dual, but it played a central role in Priddy's definition of Koszul quadratic algebras [14]. A graded augmented algebra A can be given a zero differential. Over a field \mathbf{k} and with finiteness degree-wise, the homology of the bar complex of A is the linear dual of $\text{Ext}_A(\mathbf{k}, \mathbf{k})$, compatible with their coalgebra and algebra structures. (In the case of $A = \mathbf{k}[G]$, this is reflected by Corollary 1.6 and the fact that the cohomology of BG coincides with $\text{Ext}_{\mathbf{k}[G]}(\mathbf{k}, \mathbf{k})$.) If A is a Koszul algebra, then we can replace the bar complex with a much smaller resolution, which leads to an explicit presentation of this Ext-algebra. Moreover, the theory applies to this Ext-algebra as well and replaces the cumbersome quasi-isomorphism of $A \simeq \Omega BA$ with an isomorphism $A \cong \text{Ext}_{\text{Ext}_A(\mathbf{k}, \mathbf{k})}(\mathbf{k}, \mathbf{k})$.

2. OTHER IDEAS IN THE AIR

Following up on his thesis, Serre along with Cartan considered the rational homotopy groups of a simply connected space. When shifted down, as best done by considering the homotopy groups of ΩX , those groups form a graded Lie algebra. Typically the Hurewicz homomorphism from homotopy to homology captures little information. But rationally for loop spaces, this map gives a clear picture. Building on calculations of Cartan and Serre [5], Milnor and Moore in [12] prove the following.

Theorem 2.1. *If X is simply connected, the Hurewicz map $\pi_*(\Omega X) \otimes \mathbb{Q} \rightarrow H_*(\Omega X; \mathbb{Q})$ is an injection, mapping the rational homotopy Lie algebra of X to the primitives in the Hopf algebra $H_*(\Omega X; \mathbb{Q})$.*

Another influential idea at that time was Eckmann-Hilton “Duality,” which draws attention to parallel structures in cohomology and homotopy. See the table below.

This duality is more of a philosophy than a theory. There are no theorems of the form “Given a true statement about homotopy groups, there is a true statement about cohomology groups obtained by...” or “Given a space X there is a dual space \hat{X} whose cohomology groups are the homotopy groups of X and...” Nonetheless, the duality can point to interesting directions of study. For example, looking at our table one notices a significant difference between CW structures, which are not canonical in any sense, and the Postnikov tower, which is. This leads to finding the homology decomposition of a space (see Chapter 4.H of [9] for a basic treatment).

3. QUILLEN FUNCTORS AND RATIONAL HOMOTOPY THEORY

Quillen, influenced by Kan, took the step in [15] of proving theorems not about homotopy groups but about all of homotopy theory. He must have taken Theorems 2.1 and 1.3 as an important starting point. Indeed, if the rational homology of the cobar construction computes the homology of the loop space, and one is to then take primitives to get rational homotopy groups, why not take primitives first at the level of the cobar complex itself (see exercise below)? The great advantage is that in the cobar construction one is considering the *free* associative algebra, whose primitives are known to be the free Lie algebra, so one can just use the free Lie algebra functor as a starting point. Quillen was also aware of Chevalley-Eilenberg

Cohomology	Homotopy
L.E.S of a cofibration $A \hookrightarrow X \rightarrow X/A$	L.E.S of a fibration $F \rightarrow E \rightarrow B$
Spheres and Moore spaces	Eilenberg-MacClane spaces
Suspension / desuspension	Loop space / classifying space
CW structures	Postnikov tower
Graded commutative ring structure	Graded Lie algebra structure
co- H -space (comonoid)	H -space (monoid)
pushout square / homotopy colimit	pull-back square / homotopy limit
Steenrod algebra	Stable homotopy groups of spheres
Leray-Serre spectral sequence	Blakers-Massey theorems

cohomology of Lie algebras [6], and probably knew of some cases in which applying this functor to the rational homotopy Lie algebra of a space recovered its cohomology (an easy case being wedges of spheres, whose rational homotopy Lie algebra is free). Once again, a refinement is needed, going from applying a functor at the level of algebras (in the previous case primitives, in the current case Lie algebra cohomology) to applying it at the level of chain complexes. Quillen's adaption of the Chevalley-Eilenberg construction now bears his name as well.

Quillen put these two constructions together in the following theorem.

Theorem 3.1. *The Lie algebraic cobar construction $\Omega_{\mathcal{L}ie}$ and a commutative coalgebraic bar construction B_{Comm} , which generalizes the Chevalley-Eilenberg construction, form an adjoint pair of functors*

$$DGCC \xrightleftharpoons[B_{Comm}]{\Omega_{\mathcal{L}ie}} DGLA$$

Here DGCC are 1-connected differential graded cocommutative coalgebras and DGLA are connected differential graded Lie algebras. These functors preserve all notions relevant to homotopy theory (fibrations, cofibrations, weak equivalences).

Any simply-connected space X has functorial models C_X and L_X in DGCC and DGLA respectively such that the homology of C_X is the rational homology coalgebra of X and the homology of L_X is the rational homotopy Lie algebra of X .

In current language, we would say that $\Omega_{\mathcal{L}ie}$ and B_{Comm} form a Quillen adjoint pair of functors on the model categories DGCC and DGLA, reflecting the Koszul duality of the operads $\mathcal{L}ie$ and $Comm$. This theorem gives a precise manifestation of Eckmann-Hilton duality, through the fact that these functors preserve model structures along with the symmetries of the model structure axioms. What complicates [15] significantly is that there is, to this day, no simple way to construct a commutative cochain algebra of a space and thus easily land in this picture. Quillen has to walk for forty days through the desert, producing a long chain of functors in order to produce L_X and C_X . That difficulty led Sullivan to find a simple way to produce commutativity, not on chains but on cochains. Additionally, instead of using bar

or cobar constructions Sullivan studied cofibrant replacements with some additional smallness property, the famous minimal models of [18].

Exercise 7. *Check directly in some cases that the primitives of differential graded Hopf algebra form a split sub-complex, so that the primitives of the homology of C_\bullet is isomorphic to the homology of the complex obtained by taking the primitives of C_\bullet .*

Exercise 8. *Compute the Chevalley-Eilenberg cohomology of the graded Lie algebra with three generators x, y, z in degree three with the only relation being $[x, y] = [y, z]$.*

4. KOSZUL DUALITY AND HOPF INVARIANTS

We have recently found [17] that Koszul duality and Quillen functors allow one to give a definitive treatment of rational homotopy functionals through Hopf invariants. The basic idea can be seen as using the bar complex to understand a map $f : S^n \rightarrow X$ by first passing to $\Omega f : \Omega S^n \rightarrow \Omega X$ and then evaluating cohomology classes of ΩX on the image of the fundamental class of ΩS^n . By Theorem 2.1, such invariants are complete.

We must pause to make a choice in notation. If one is studying the cohomology of ΩX using the cochains on X , one could either denote the construction you use by Ω to reflect topology or B to denote a bar construction which is applied to an algebra (rather than a cobar construction applied to a coalgebra). The algebraists seem to have won this notational conflict, so we consider $BC^*(X)$, the bar construction on the cochains of X with their associative cup product. We let $H_B^*(X)$ denote the homology of $BC^*(X)$. Define the weight of a generator of a bar complex to be the number of elements appearing.

Lemma 4.1. *$H_B^{n-1}(S^n)$ is rank one, generated by an element of weight one corresponding to a generator of $H^n(S^n)$.*

Exercise 9. *Prove this. Hint: you'll need the Künneth theorem to put yourself in a position to do some "weight reduction," as we use below.*

Definition 4.2. Let $\gamma \in B^{n-1}(C^*S^n)$ be a cocycle. Define $\tau(\gamma) \simeq \gamma$ to be a choice of weight one cocycle to which γ is cohomologous.

Define $\int_{B(S^n)}$ to be the map from cocycles in $B^{n-1}(C^*S^n)$ to \mathbb{Z} given by $\int_{B(S^n)} \gamma = \int_{S^n} \tau(\gamma)$, where \int_{S^n} denotes evaluation on the fundamental class of S^n .

Define $\eta(\gamma)$, the Hopf invariant associated to γ by $\eta(\gamma)(f) = \int_{B(S^n)} f^* \gamma$.

The choice of Hopf cochain is not unique, but the corresponding Hopf invariants are. It is immediate that the Hopf invariants are functorial. Moreover, note that the definitions hold with any ring coefficients. Topologically we have the following interpretation.

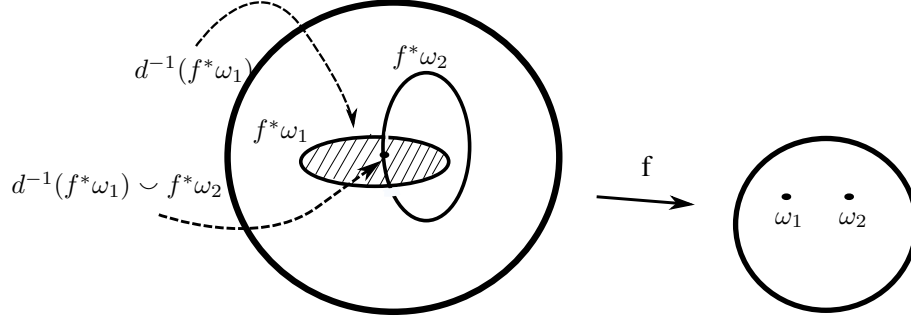
Proposition 4.3. *$\eta(\gamma)(f)$ coincides with the evaluation of the cohomology class given by γ in $H^{n-1}(\Omega X)$ on the image under Ωf of the fundamental class in $H_{n-1}(\Omega S^n)$.*

4.1. Examples.

Example 4.4. A cocycle of weight one in $B(C^*X)$ is just a closed cochain on X , which may be pulled back and immediately evaluated. Decomposable elements of weight one in $B(X)$ are null-homologous, consistent with the fact that products evaluate trivially on the Hurewicz homomorphism.

Example 4.5. Let ω_1 and ω_2 be generating 2-cocycles on S^2 and $f : S^3 \rightarrow S^2$. Then $\gamma = -|\omega_1|\omega_2|$ is a cocycle in $B(C^*S^2)$ which f pulls back to $-|f^*\omega_1|f^*\omega_2|$, a weight two cocycle of total degree two on S^3 . Because $f^*\omega_1$ is closed and of degree two on S^3 , it is exact. Let $d^{-1}f^*\omega_1$ be a choice of a cobounding cochain. Then

$$d(|d^{-1}f^*\omega_1|f^*\omega_2|) = |f^*\omega_1|f^*\omega_2| + |d^{-1}f^*\omega_1 \smile f^*\omega_2|.$$



Whitehead's integral, viewed through intersections of supports of cochains.

Thus $f^*\gamma$ is homologous to $|d^{-1}f^*\omega_1 \smile f^*\omega_2|$, and the corresponding Hopf invariant is $\int_{S^3} d^{-1}f^*\omega_1 \smile f^*\omega_2$, which when choosing $\omega_1 = \omega_2$ is the classical formula for Hopf invariant given by Whitehead [19]. It is a direct translation of the linking number definition of Hopf invariant into the language of cochains. Understanding this formula from the point of view of the bar construction has, to our knowledge, only come over fifty years since all of these concepts were introduced.

Example 4.6. For an arbitrary X and cochains x_i, y_i and θ on X with $dx_i = dy_i = 0$ and $d\theta = \sum (-1)^{|x_i|} x_i \smile y_i$, the cochain $\gamma = \sum |x_i|y_i| + |\theta| \in B(C^*X)$ is closed. The possible formulae for the Hopf invariant are all of the form

$$\int_{S^n} \left(f^*\theta - \sum \left((-1)^{|x_i|} t_i \cdot d^{-1}f^*x_i \smile f^*y_i + (1 - t_i) \cdot f^*x_i \smile d^{-1}f^*y_i \right) \right),$$

for some real numbers t_i . This generalizes formulae given in [18], [7] and [8].

By choosing $t = \frac{1}{2}$ we see that reversing the order to consider $\sum |y_i|x_i|$ will yield the same Hopf invariant, up to sign. Thus $\sum |x_i|y_i| \mp |y_i|x_i|$ yields a zero Hopf invariant. There are many Hopf invariants of the classical bar construction which are zero, a defect remedied by using the Lie coalgebra cobar construction.

Exercise 10. Suppose x and y are cochains supported on codimension two submanifolds X and Y of W and θ satisfies $d\theta = x \smile y$ and is supported on a codimension three submanifold which cobounds $X \cap Y$. Draw pictures of how the Hopf invariant associated to $|x|y| \mp |\theta|$ evaluates some map $f : S^3 \rightarrow X$. Moreover, draw pictures of what can happen in $S^3 \times \mathbb{I}$ if one has a homotopy between f and g . [Hint: Start with the picture in the figure, but then draw in the preimage of the support of θ ; then, think about what can happen with the preimage of $X \cap Y$ through a homotopy.]

One can do similar calculations in higher weight, and interpret them all when one chooses cochains supported on submanifolds in terms of linking behavior of the preimages of those submanifolds. See [17].

4.2. The cokernel and kernel of the Hopf invariant map. Our Hopf invariant construction defines a homomorphism $\eta : H_*(B(C^*(X; \mathbb{Z}))) \rightarrow \text{Hom}(\pi_*X, \mathbb{Z})$. It follows from Proposition 4.3 and Theorem 2.1 that this map is surjective when tensored with the rational numbers, and thus is full rank.

Problem 11. Compute the cokernel of η . By Adams' celebrated result [1], this cokernel is trivial for X an odd sphere and for S^2, S^4 and S^8 , and it is $\mathbb{Z}/2$ for other even spheres.

The proofs in [17] show that one might be able to directly understand the relation of this cokernel to lack of commutativity of cup product. Though this cokernel is clearly a very subtle homotopy invariant, we do not see any applications of its calculation.

Also, η has a very large kernel, explained from the operadic viewpoint as the fact that we are taking the wrong bar construction. The rational PL cochains on a simplicial set are commutative, so we should be

taking a bar construction over the Koszul dual cooperad, namely the Lie cooperad, rather than associative cooperad. The homology of such a bar construction $B_{\mathcal{L}ie}$ is known as Harrison homology. Using a graphical model for the Lie cooperad developed in [16] which makes calculations possible, we prove the following.

Theorem 4.7. [17] *There is a Hopf invariant map $\eta^{\mathcal{L}ie}$ which factors the map η such that $\eta^{\mathcal{L}ie} : H_{B_{\mathcal{L}ie}}^*(X) \rightarrow \text{Hom}(\pi_*(X), \mathbb{Q})$ is an isomorphism of Lie coalgebras.*

It is almost immediate that similar Hopf invariants can be used to concretely realize similar isomorphisms arising for Koszul pairs in general.

To summarize, in homology theory it has been helpful to have geometry attached not only to homology but cohomology. In particular, homology classes are often represented by closed submanifolds and cohomology classes are represented by either forms or proper submanifolds. The geometry of homotopy groups arising from their definition is almost too simple. To reflect on the geometry of Theorem 2.1, we notice that the Lie algebra generators of $\pi_*(X) \otimes \mathbb{Q}$ have non-trivial Hurewicz image, as noticed by Cartan-Serre. That is, the rational homotopy groups of X are spanned by Whitehead products of spherical homology classes. Our work on Hopf invariants shows that the geometry of homotopy functionals is given by linking invariants, as perfectly governed by the Lie cooperad, completing the geometric understanding of these basic functors in the rational setting. We hope these ideas can be extended to the non-simply connected setting, and perhaps - at least in part - in characteristic p .

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